MAL 122 : REAL AND COMPLEX ANALYSIS

Department of Mathematics, IIT Delhi

MAJOR 05 - 05 - 2007

Max. Marks : 50

ALL QUESTIONS ARE COMPULSORY AND CARRY EQUAL MARKS

1. State whether the following are True or False. Justify your answer.
   (a) Continuous image of a complete metric space is complete.
   (b) A Cauchy sequence $< x_n >$ in a metric space (X, d) converges if it has a cluster point.

2. Let (X, d) be a metric space and let $x_0 \in X$. Show that the function $f : X \to \mathbb{R}$ defined as $f(x) = d(x, x_0)$ is continuous. Is it uniformly continuous? Justify.

3. State whether the following are True or False. Justify your answer.
   (a) An analytic function $f : \mathbb{C} \to \mathbb{C}$ is constant if and only if $\overline{f}$ is analytic.
   (b) If v_1 and v_2 are harmonic conjugates of u in a domain D, then v_1 and v_2 must differ by an additive constant.

4. Let $f : D \to \mathbb{C}$ be an analytic function on a domain D, and let $z_0 \in D$. Let γ be the (positively oriented) boundary of $B(z_0, r)$ lying within D. Prove that

$$f(z) - f(z_0) - \frac{z - z_0}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^2} dw = \frac{(z - z_0)^2}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^2} dw$$

for any $z \in B(z_0, r)$. Deduce that

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^2} dw$$

5. Evaluate $\int_\gamma \frac{e^z}{z-1} dz$ and $\int_\gamma \frac{e^z}{\pi i - 2z} dz$ where $\gamma = \{z \in \mathbb{C} : |z| = 2\}$.

6. Let f be an entire function such that for each $z_0 \in \mathbb{C}$ atleast one coefficient in the expansion $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(z) = z^2$ for all $z \in \mathbb{Q}$. Does it follow that $f(z) = z^2$ for all $z \in \mathbb{C}$? Justify.

8. Let f be an entire function such that $|f(z)| \le C|z|$ for all z, where C is a fixed positive number. Show that $f(z) = az$ where a is a complex constant.

9. Find the Laurent expansion of the function $f(z) = \frac{1}{(z+2)(z^2+1)}$ about the point z = i, valid in the annulus $2 < |z - i| < \sqrt{5}$.

10. Evaluate $\int_0^\infty \frac{sinx}{x} dx$.