PROF - AMITABHA TRIPATHI
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Department of Mathematics

MAL 122: Introduction to Real & Complex Analysis

2022-23: Semester II

Major Exam

7 May 2023

You may attempt *all* questions. Please begin each answer on a new page, and give adequate explanation for full credit.

1. Let $\{q_1, q_2, q_3, \ldots\}$ denote the set of rationals in $[0,1]$. For $x \in [0,1]$, let $A_x = \{n \in \mathbb{N} : q_n \leq x\}$. Define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \sum_{n \in A_x} \frac{1}{2^n}.$$
Prove that f restricted to the irrationals in $[0,1]$ is continuous. [4]

2. Let (X,d) be a metric space such that every real-valued continuous function on X is uniformly continuous. Show that X is a complete metric space. [4]

3. Show that a metric space (X,d) is a Baire space if and only if the complement of every meager set is dense in X. [4]

4. Let (X,d) be a metric space. If A, B are closed subsets of X such that $A \cup B$ and $A \cap B$ are both connected, show that A and B are both connected. [4]

5. Let $0 < R_1 < R_2$, and let $\mathscr{D} := \{z : R_1 < |z| < R_2\}$. If $u(x,y) = \frac{1}{2}\,\mathrm{Log}\,(x^2+y^2)$ in $\mathscr{D}$, determine if there exists a function $v(x,y)$ in $\mathscr{D}$ such that $f = u + iv$ is analytic in $\mathscr{D}$. **Justify your answer.** [4]

6. Suppose f is entire and satisfies
$$|f(z)| \leq \frac{1}{|\Im m\, z|}$$
for all $z \in \mathbb{C}$. By estimating $|f|$ on the circle $|z| = R$ via the function $g(z) := (z^2 - R^2)f(z)$, prove that $f(z) = 0$ for all $z \in \mathbb{C}$. [4]

7. Use a contour integral to show that
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}\, dx = \frac{1}{2}\pi(1 - e^{-2}).$$
[6]

8. If f is entire and if, for some integer $n \geq 0$, there exist positive constants A and B such that
$$|f(z)| \leq A + B|z|^n$$
for each $z \in \mathbb{C}$, prove that f is a polynomial of degree at most n. [4]

9. Determine all entire functions f that satisfy $|f(z)| \leq e^{\Re e\, z}$ for each $z \in \mathbb{C}$. [4]

10. Let n be a positive integer, R a positive real number, and $a \in \mathbb{C}$ be such that $|a| > e^R/R^n$. Prove that the equation $az^n - e^z = 0$ has n solutions satisfying $|z| < R$. If $|R| = 1$, show that these solutions are simple roots. [6]