MINOR EXAMINATION MTL122

MAXIMUM MARKS: 30
DURATION: 1 hour 30 minutes

Instructions: Answer as much as you can. Maximum you can obtain is 30 marks. Whenever you refer to any of the results done in classes/tutorials, clearly state it. Marks will NOT be awarded for incomplete answers.

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(1) Let $X \neq \emptyset$. Consider the map $d : X \times X \to \mathbb{R}$ such that
   (a) $d(x, x) = 0$ for all $x \in X$,
   (b) $d(x, y) = d(y, x)$ for all $x \neq y$,
   (c) $d(x, y) \in [1, 2]$ for all $x \neq y$.
   Show that (X, d) is a metric space. 2 marks

(2) Let (X, d) be a metric space and $x \neq y$ be two points of X. Does there exist two open sets U, V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$? Justify your answer. 2 marks

(3) Let (X, d) be a metric space such that every function $f : X \to \mathbb{R}$ is continuous. Show that d is discrete metric on X. 3 marks

(4) Consider the Euclidean metric space $(\mathbb{R}^2, d_2)$. Let U be an open subset of $\mathbb{R}^2$. Show that U can be written as union of sets of the form $(a, b) \times (c, d)$ where $(a, b), (c, d)$ are open intervals with $a, b, c, d \in \mathbb{R}$. 4 marks

(5) Let X be the set of all sequences of real numbers. For two sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$, define

$$d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) := \sum_{n\geq 1} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Show that (X, d) is a metric space. 4 marks

(6) Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be an uniformly continuous function such that $|f(x)| \geq \delta$ for all $x \in A$ for some $\delta > 0$. Is $1/f$ uniformly continuous on A? Justify your answer. 4 marks

(7) Let (X, d) be a metric space. A subset A of (X, d) is called nowhere dense if $\mathrm{Int}(\overline{A}) = \emptyset$. Prove that a closed set in X is nowhere dense if and only if $X \setminus A$ is dense. 4 marks

(8) Let (X, d) be a metric space. Prove that a subset A of (X, d) is closed if and only if its intersection with every compact subset of (X, d) is closed. 4 marks

(9) Let (X, d) be a metric space where closure of every open set is open. Show that d is the discrete metric on X. 4 marks

(10) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{1}{n^3(x - \frac{1}{n})^2 + 1} \text{ for all } x \in \mathbb{R},$$

be a sequence of functions. Find out the limit function. Is the convergence uniform? Justify your answer. 4 marks