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# Department of Mathematics
## MAL 122: Introduction to Real & Complex Analysis
## 2022-23: Semester II
## Minor 2

*26 March 2023*

**Attempt *any* five questions. Each question is worth *five* points. Please begin each answer on a new page, and give adequate explanation for full credit.**

1. Let $(X, d)$ be a compact metric space, and let $f : X \to X$ be such that $d(f(x_1), f(x_2)) < d(x_1, x_2)$ whenever $x_1 \neq x_2$.
   (a) Show that $g(x) := d(x, f(x))$ attains its *minimum* on $X$. [$2\frac{1}{2}$]
   (b) Prove that $f$ has exactly one fixed point. [$2\frac{1}{2}$]

2. (a) Show that every compact metric space is separable. [$2\frac{1}{2}$]
   (b) Consider the metric space $\mathbb{Q}$ with Euclidean distance. Let $S = (a, b) \cap \mathbb{Q}$, where $a, b \in \mathbb{R} \setminus \mathbb{Q}$. Discuss whether or not $S$ is a compact subset of $\mathbb{Q}$. [$2\frac{1}{2}$]

3. Let $(X, d)$ be a metric space.
   (a) Prove that $X$ is connected if and only if every *continuous* function $f : X \to \{0, 1\}$ is constant. [$2\frac{1}{2}$]
   (b) If $A$ be a connected subset of $X$ and if $A \subseteq B \subseteq \overline{A}$, use the result in part (a) to prove that $B$ is also connected. [$2\frac{1}{2}$]

4. (a) Let $(X, d)$ be a metric space in which any two point $x_1$ and $x_2$ is contained in some connected subset $A$ of $X$. Prove that $X$ is connected. [2]
   (b) Determine whether or not the set $\{(x, y) : x^2 + y^2 = 1\}$ is a connected subset of $\mathbb{R}^2$. Justify your answer. [3]

5. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) Show that $f$ satisfies the *Cauchy-Riemann* equations at $(0, 0)$. [2]
   (b) Show that $f'(0)$ does not exist. [2]
   (c) Explain why $f$ fails to have a derivative at $z = 0$. [1]

6. (a) Under stereographic projection, deduce the expression to correspond points $(x_1, x_2, x_3)$ on the Riemann sphere $x_1^2 + x_2^2 + x_3^2 = 1$ to points $(x, y)$ on the $XY$-plane both ways. [2]
   (b) Show that $z$ and $z'$ correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z'} = -1$. [3]