INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
SEMESTER I, 2021 − 22
MTL 180 (DISCRETE MATHEMATICAL STRUCTURES)
MAJOR EXAMINATION
DURATION: 9:00 AM - 11:30 AM

Instructor: Biplab Basak — Date: 16/11/2021

FULL MARKS: 47

QUESTION 1: There were 2 friends Avi and Shivam, and Shivam was stuck behind a door in a mystery room. In the mystery room, there were 3 doors and Avi had to guess the door in which Shivam was behind. Avi has only 1 chance to retrieve his friend and hence wants to be 100% sure before making any decision. Each door has some clue written on it; the clues are:
Door 1: "Shivam is not here"
Door 2: "Shivam is not here"
Door 3: "Shivam is in Door 2"
Only one message is true; the other two are false. Behind which door Shivam is there? Formalize the puzzle in Propositional Logic and find the solution using a truth table. (6)

QUESTION 2: Let $X = [-1, 1] \times [0, 1)$ and ρ be a relation defined on X by $(a, b)\rho(c, d)$ if and only if $a + b^2 = c + d^2$, for (a, b), $(c, d) \in X$.

- (i) Find the set $\mathcal{P}$ of all equivalence classes corresponding to ρ.
- (ii) Define a relation $\prec$ on $\mathcal{P}$ in the following way: for $A, B \in \mathcal{P}$, $A \prec B$ iff $a+b^2 < c+d^2$, for some $(a, b) \in A$ and $(c, d) \in B$. Prove that $\prec$ is an order relation.
- (iii) Assume that $\mathbb{R}$ has the $l.u.b$ property. Does the set $\mathcal{P}$ with the order $\prec$ have the $l.u.b$ property? (2+3+3)

QUESTION 3: Find the generating function for the solutions to $a_n - a_{n-1} - 2a_{n-2} = 2^n$ for $n \geq 2$, and $a_0 = 2, a_1 = 1$. Then using that, find a formula for a_n. (7)

QUESTION 4: Let G and complement of G both be connected simple planer graphs. Let n, e be the number of vertices and number of edges of G respectively. Discuss whether the following values of the pair (n, e) are possible or not.
$(i)\,(n, e) = (6, 8)$; $(ii)\,(n, e) = (7, 11)$; $(iii)\,(n, e) = (8, 19)$; $(iv)\,(n, e) = (10, 20)$; $(v)\,(n, e) = (11, 26)$. Either construct the graph when it is possible or show that the value of the pair (n, e) is not possible. (7)

QUESTION 5: Answer the following questions. (2+2+3)

- (i) Let G be a graph with chromatic number d. Let m be the number of edges. Prove that $m \geq \frac{d(d-1)}{2}$.
- (ii) Either prove or disprove that a graph with chromatic number 3 has the subgraph K_3.

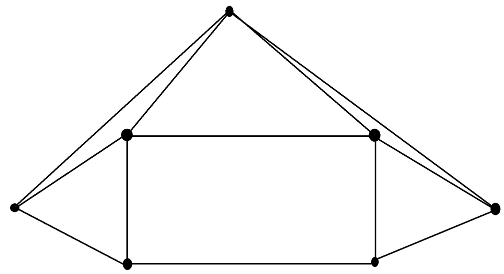


Figure 1: The graph G of Question 5

(iii) Find chromatic number and clique number for the above graph G (in Figure 1) and its complement G^c.

QUESTION 6: Consider the diagram (land - Deep Gray; river - Light Gray; bridge - Black) of a city.

(i) Is it possible to take a walking tour of the city that crosses each of the eleven bridges over the river exactly once?

(ii) Is it possible to start at some place and take a walk that uses each of the eleven bridges exactly once, and ends at the starting place?

In each of the above cases, if such walk is possible then draw/mention the walk. (6)

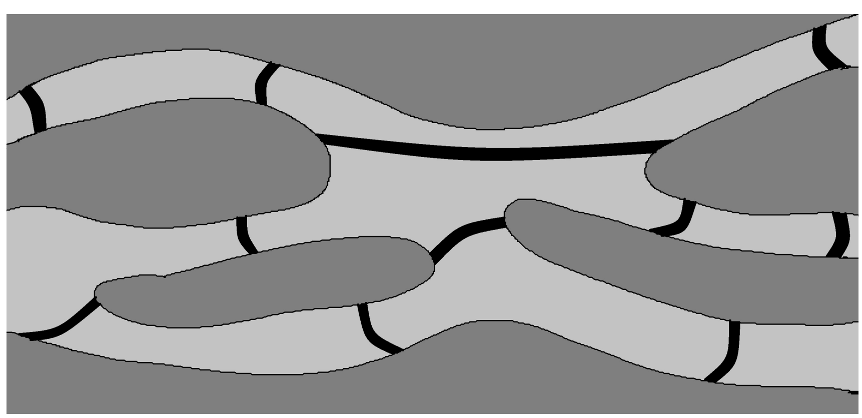


Figure 2: The diagram of the city of Question 6

QUESTION 7: Let $P = \{1, 2, \ldots, 100\}$. Let X and Y be the set of all subsets of P of cardinality 49 and 50 respectively. Consider the graph $G := (X, Y, E)$ with bipartition X and Y, and with edge set $E = \{AB \mid A \in X, B \in Y \text{ and } A \subset B\}$. What is the size of the maximum matching in G? (6)

The End

Solutions

Solution 1:

Let B_i with i $\in$ {1,2,3} stand for "Shivam is in the i-th door". We can formalize the statements of the problem as follows:

1) One door contains Shivam and the other two are empty.

$$(B_1 \wedge -B_2 \wedge -B_3) \vee (-B_1 \wedge B_2 \wedge -B_3) \vee (-B_1 \wedge -B_2 \wedge B_3) \quad (1)$$

2) Only one message is true; the other two are false.

$$(-B_1 \wedge - - B_2 \wedge -B_2) \vee (- - B_1 \wedge -B_2 \wedge -B_2) \vee (- - B_1 \wedge - - B_2 \wedge B_2) \quad (2)$$

(2) is equivalent to:

$$(B_1 \wedge -B_2) \vee (B_1 \wedge B_2) \quad (3)$$

Now, let's compute the truth table for (1) and (3) article array

B_1	B_2	B_3	Expression (1)	Expression (3)
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	F	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

The only instance in the table which satisfies both the expressions is the one with V(B_1) = T, V(B_2) = F and V(B_3) = F, which implies that Shivam is behind Door 1.

Solution of (2):

(i) Take $(a, b) \in X$ and $a + b^2 = r$. Then $-1 \leq r < 2$. For each such r consider the set $A_r = \{(a, b) \in X : a + b^2 = r\}$ this A_r is the equivalence class $cl(a, b)$. And the set $\mathcal{P} = \{A_r : -1 \leq r < 2\}$.

(ii) **Comparability:** Take $A \neq B \in \mathcal{P}$ and suppose $(a, b) \in A$, $(c, d) \in B$ then either $a + b^2 <$ or $> c + d^2$ and we will have $A \prec B$ or $B \prec A$ accordingly.

Nonreflexivity: For any $A \in \mathcal{P}$ there is no $(a, b), (c, d) \in A$ such that $a + b^2 < c + d^2$ holds. So $A \prec A$ does not holds.

Transitivity: Take $A, B, C \in \mathcal{P}$ and $A \prec B$, $B \prec C$ holds. Then there are $(a, b) \in A$, $(c, d) \in B$ and $(e, f) \in C$ such that $a + b^2 < c + d^2$ and $c + d^2 < e + f^2$ holds and therefore $a + b^2 < e + f^2$. Thus $A \prec C$.

(iii) Let $\mathcal{A}$ be a non empty subset of $\mathcal{P}$ which is bounded above by $A_r \in \mathcal{P}$. Then by the definition $r < 2$. Consider the set $S = \{s \in [-1, 2) : A_s \in \mathcal{A}\}$. Since $A_s \prec A_r$, $s < r$ for any $s \in S$. Therefore S is a non empty subset of $\mathbb{R}$ which is bounded above. By $l.u.b$ property of $\mathbb{R}$ there is a least upper bound for S say r_0. Then $-1 \leq r_0 < 2$ and A_{r_0} is the $l.u.b$ for $\mathcal{A}$ because if $A_t \prec A_{r_0}$ is an upper bound for $\mathcal{A}$ then we must have $t < r_0$ and t is an upper bound for S.

Solution of Q.3: Consider the following non-homogeneous recurrence:

$$b_n - b_{n-1} - 2b_{n-2} = 2^n \quad (\text{for } n \geq 2), \qquad b_0 = 2, \quad b_1 = 1.$$

Multiplying this recurrence with x^n for each $n \geq 2$ and adding yields a sum in which we can rewrite the terms:

$$\underbrace{\sum_{n=2}^{\infty} b_n x^n}_{F(x) - b_0 - x b_1} - \underbrace{\sum_{n=2}^{\infty} b_{n-1} x^n}_{x \cdot (F(x) - b_0)} - 2 \underbrace{\sum_{n=2}^{\infty} b_{n-2} x^n}_{x^2 F(x)} = \underbrace{\sum_{n=2}^{\infty} 2^n x^n}_{\frac{1}{1-2x} - 1 - 2x} .$$

Solving for $F(x)$ yields:

$$F(x)(1 - x - 2x^2) = \frac{1}{1 - 2x} + 1 - 3x$$

$$\Rightarrow F(x) = \frac{2 - 5x + 6x^2}{(1 - 2x)(1 - x - 2x^2)}$$

From here, apply partial fraction decomposition with a method of your choice, to get

$$F(x) = -\tfrac{1}{9} \frac{1}{1 - 2x} + \tfrac{2}{3} \frac{1}{(1 - 2x)^2} + \tfrac{13}{9} \frac{1}{1 + x}.$$

From this we can obtain the coefficients again using identities we know.

$$F(x) = -\tfrac{1}{9} \sum_{n=0}^{\infty} 2^n x^n + \tfrac{2}{3} \left(\sum_{n=0}^{\infty} 2^n x^n \right)^2 + \tfrac{13}{9} \sum_{n=0}^{\infty} (-1)^n x^n.$$

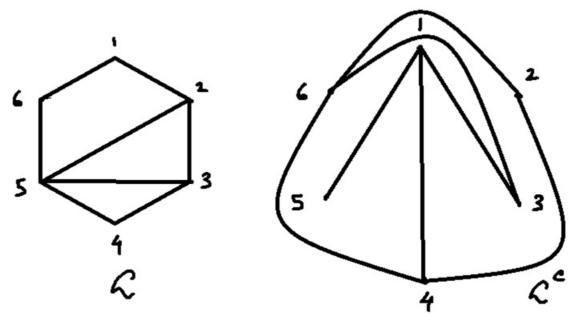
$$= -\tfrac{1}{9} \sum_{n=0}^{\infty} 2^n x^n + \tfrac{2}{3} \sum_{n=0}^{\infty} (n + 1) 2^n x^n + \tfrac{13}{9} \sum_{n=0}^{\infty} (-1)^n x^n.$$

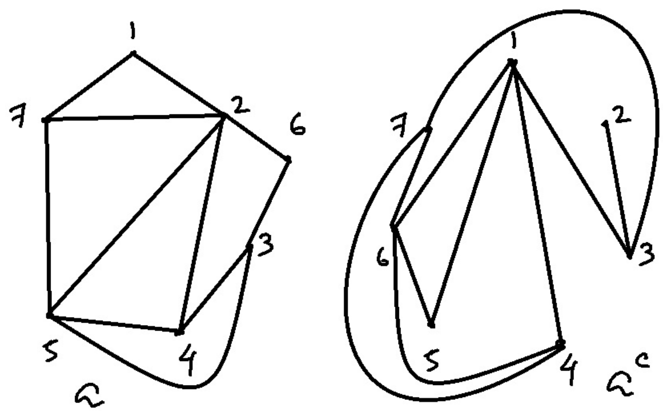
The coefficients are therefore:

$$b_n = -\tfrac{1}{9} 2^n + \tfrac{2}{3}(n + 1) 2^n + \tfrac{13}{9}(-1)^n.$$

Solution of (4):

(i) & (ii) Given that G and complement of G both are connected simple planer graphs. Let n, e be the number of vertices and number of edges of G respectively. One of such G and G^c are given in the following figure.





(iii) A connected planner graph with n vertices must have edges $e \leq 3n-6$. Here $e = 19$ and $n = 8$, thus $e = 19 > 18 = 3n-6$. Thus G is not planer. Therefore, this choice for (n, e) is not possible.

(iv) A connected planner graph with n vertices must have edges $e \leq 3n-6$. Here $e = 20$ and $n = 10$. Thus, G^c has 10 vertices and $\binom{10}{2} - 20 = 25$ edges which is more than $24 = 3n-6$. Thus G^c is not planer. Therefore, this choice for (n, e) is not possible

(v) A connected planner graph with n vertices must have edges $e \leq 3n-6$. Here $e = 26$ and $n = 11$. Thus, G^c has 11 vertices and $\binom{11}{2} - 26 = 29$ edges which is more than $27 = 3n-6$. Thus G^c is not planer. Therefore, this choice for (n, e) is not possible

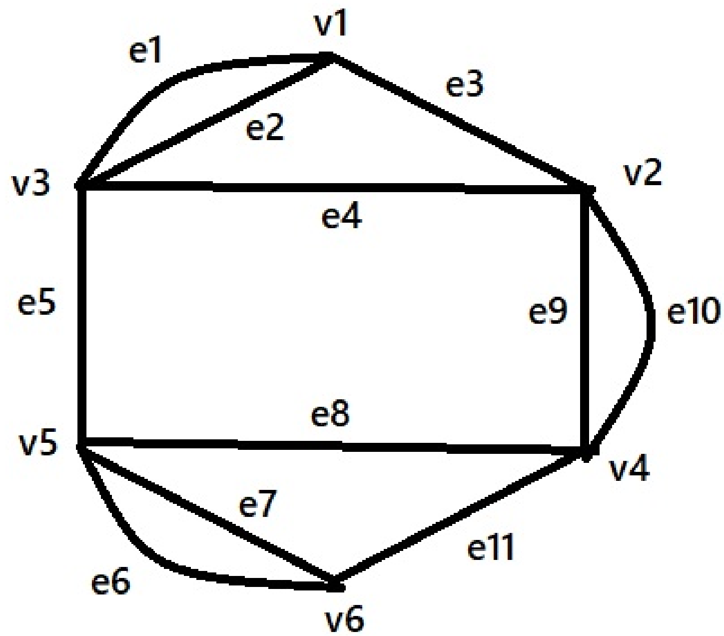
Solution of (5):

(i) Since the graph has chromatic number d, it has a proper coloring with exactly d colors, say $1, 2, \ldots, d$. Let V_i be the set of vertices with color i, for $1 \leq i \leq d$. There must be an edge between some vertices of V_i with some vertices of V_j, for $1 \leq i \neq j \leq d$, otherwise we can color the vertices of V_i and V_j with same color and chromatic number will be less than d. Therefore, we must have at least $\binom{d}{2} = \frac{d(d-1)}{2}$ edges.

(ii) This is not true. We can take a 5-cycle C_5. It has chromatic number 3 but does not have a subgraph K_3.

(iii) The chromatic number and clique number of G are 4 and 3 respectively. The chromatic number and clique number of G^c are 3 and 2 respectively.

Solution of (6): First we translate the bridge problem into a graph theory problem. For each connected component of land, we take a vertex, i.e., we have 6 vertices. Then two vertices will have an edge, if the corresponding land is connected by a bridge. Thus we get a multigraph, with 6 vertices and 11 edges.



w=v1e1v3e2v1e3v2e4v3e5v5e6v6e7v5e8v4e9v2e10v4e11v6

We know that 'A connected graph G is an Eulerian graph (i.e., G has a closed walk which travels each of the edges exactly once) if and only if all vertices of G are of even degree.' From this we can also say that, if a graph G has exactly two vertices (say u and v) of odd degree and remaining vertices have even degree, then there is a walk in G from u to v, which travels each of the edges exactly once.

(i) Since the graph G has exactly two vertices v_1 and v_6 of odd degree, there is a walk in G from v_1 to v_6, which travels each of the edges exactly once. Thus, it is possible to take a walking tour of the city that crosses each of the eleven bridges over the river exactly once.

(ii) Since G has two vertices of odd degree, G is not an Eulerian graph. Thus, it is NOT possible to start at some place and take a walk that uses each of the eleven bridges exactly once, and ends at the starting place.

The walk w in G from v_1 to v_6, is given in the figure as the sequence of edges $e_1, \ldots, e_{11}$, i.e., $v_1e_1v_3e_2v_1e_3v_2e_4v_3e_5v_5e_6v_6e_7v_5e_8v_4e_9v_2e_{10}v_4e_{11}v_6$

Solution of (7): Any vertex of X has degree 51 whereas every vertex of X has degree 50 in G. Take any set $S \subseteq X$ then total no. of edges between S and $\Gamma(S)$ is $51|(S)|$ and every such edge incident to a vertex of Y since G is bipartite. Since any vertex of $\Gamma(S)$ incident to only 50 edges thus $\Gamma(S)$ must contains at least $|S|$ vertices. Therefore, $|\Gamma(S)| \geq |(S)|$. From Hall's Marriage theorem there is a matching from X to Y which covers X. Thus the the maximum matching in G has size $\binom{100}{49}$.