Department of Mathematics
MTL 180: Discrete Mathematical Structures
2020-2021: Semester I
Major Exam

6 January 2021

You may attempt *all* questions, beginning each answer on a new sheet. Explain your answer in sufficient detail. Maximum marks for this paper is 50.

1. Let $f : X \to Y$ and $g : Y \to X$ be one-one mappings. Show in detail how one may construct a one-one, onto mapping from X to Y, and justify why this mapping is a bijection. **[6]**

2. (a) Let $n \in \mathbb{N}$. Determine the set of all integers which leave a remainder $n+1, n+2, n+3$ when divided by $2n+1, 2n+3, 2n+5$ respectively. **[3]**

   (b) Evaluate the sum $\sum_{d|n}(-1)^{n/d}\phi(d)$. **[4]**

3. Let G be a *finite* group, and let φ be an automorphism of G such that $\varphi(x) = x$ if and only if $x = e$.

   (a) Prove that every $g \in G$ can be represented as $g = x^{-1}\varphi(x)$ for some $x \in G$. **[3]**

   (b) Moreover, if $\varphi\big(\varphi(x)\big) = x$ for every $x \in G$, prove that G is abelian. **[3]**

4. (a) Give a *combinatorial* proof for the identity $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$. **[3]**

   (b) Show that the Stirling number of the second kind $S(n, n-2)$ equals $\binom{n}{3} + 3\binom{n}{4}$. **[3]**

5. (a) Let $a_1, \dots, a_k$ be integers none of which is a multiple of m. If $2k > m$, prove that $m \mid (a_i - a_j)$ or $m \mid (a_i + a_j)$ for some $i \neq j$. **[3]**

   (b) Use the `Principle of Inclusion & Exclusion` to evaluate the sum

   $$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n-i}{r}$$

   for $m \leq r \leq n$. **[3]**

6. (a) Let S_n denote the sum of the *greatest odd* divisors of $1, \dots, 2^n$. Set up a first-order recurrence for $\{S_n\}$, and evaluate S_n. **[3]**

   (b) Determine the sequence whose generating function is $(z-2)/(z^2+z-1)$. **[3]**

7. (a) Prove that there does *not* exist a (simple) graph of order n with degree set $\{1, n-2\}$ if n is an *even* integer > 4. **[3]**

   (b) Let G be an r-regular graph with length of smallest cycle 4. Prove that G has *at least* $2r$ vertices, and determine all such graphs with $2r$ vertices. **[3]**

8. Let G be an n-vertex graph, where $n \geq 3$. If G has m edges and $m \geq \frac{1}{2}(n^2 - 3n + 6)$, prove that G is *hamiltonian*. Give an example of a non-hamiltonian graph with $m = \frac{1}{2}(n^2 - 3n + 4)$ edges. **[4+2]**

9. (a) Let G be a *connected plane* k-regular graph in which each face is bounded by a cycle of length ℓ. Show that $\frac{1}{k} + \frac{1}{\ell} > \frac{1}{2}$. **[3]**

   (b) Determine all graphs G for which the chromatic polynomial $\chi(G; k)$ equals $k(k-1)^{n-1}$. **[3]**