Department of Mathematics
MTL 180: Discrete Mathematical Structure
2017-2018: Semester I
Minor Exam 1

20 September 2017

You may attempt *any five* questions. Each question is worth *five* marks. Explain your answer in sufficient detail.

1. (a) Let $f : A \to B$. If $\{B_1, \ldots, B_n\}$ is a partition of B, prove that $\{f^{-1}(B_1), \ldots, f^{-1}(B_n)\}$ is a partition of A. [2]

   (b) Let $S = \{1, \ldots, n\}$, and let A be a subset of S. Define a relation $\mathscr{R}$ on the set of all subsets $\mathcal{P}(S)$ of S by

   $$X \mathscr{R} Y \iff A \cap X = A \cap Y.$$

   Determine the number of equivalence classes in the partition induced by $\mathscr{R}$. [3]

2. (a) Suppose $(\mathscr{P}, \preceq)$ is a poset, and let $\mathscr{L} = \{a, b\} \subset \mathscr{P}$, with $a \preceq b$. Let $\mathscr{Q} = \mathscr{P} \times \mathscr{L}$, and let $\mathscr{A}$ be any antichain in $\mathscr{Q}$. Let $\mathscr{B}$ be the largest possible subset of $\mathscr{P}$ such that $\mathscr{B}$ does not contain a chain of length of size exceeding 2. Show that $|\mathscr{A}| \leq |\mathscr{B}|$. [$2\frac{1}{2}$]

   (b) Prove that any b in a Boolean lattice, $b \neq \mathbf{0}$, can be expressed as a join of atoms. **You need not show that this expression is unique.** [$2\frac{1}{2}$]

3. Show that the set $\mathbb{R}^{\mathbb{R}}$ of all real-valued functions defined on $\mathbb{R}$ is not numerically equivalent to $\mathbb{R}$ by showing the nonexistence of a surjection from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{R}$. Provide sufficient details. [5]

4. (a) Let $X = \{0, 1\}$. Prove or disprove the equivalence of

   $$\exists!\, x \in X, P(x) \text{ and } \big(P(0) \wedge (\neg P(1))\big) \vee \big(P(1) \wedge (\neg P(0))\big).$$

   [3]

   (b) Comment on the following proof of the statement "*Any set of horses are all of the same colour*" by induction.

   Basis of induction: The statement is trivially true for one horse.
   Induction step: Suppose the statement holds for n horses, and we have $n+1$ horses, $H_1, \ldots, H_{n+1}$. By induction hypothesis, the n horses $H_1, \ldots, H_n$ are all the same colour, as are the n horses $H_2, \ldots, H_{n+1}$. Hence the horses $H_1, \ldots, H_{n+1}$ are also of the same colour.

   [2]

5. (a) If $n > 1$ is an integer not of the form $6k + 3$, prove that $n^2 + 2^n$ is composite. [2]

   (b) If m, n are positive integers, prove that $\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1$. [3]

6. (a) Prove that if p and $p + 2$ are both primes, then $p(p+2) \mid \left[4\big((p-1)! + 1\big) + p\right]$. [2]

   (b) Show that 13 is the *only* prime that divides two successive integers of the form $n^2 + 3$. [3]