INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
SEMESTER I, 2021 – 22
MTL 180 (DISCRETE MATHEMATICAL STRUCTURES)
MINOR EXAMINATION
DURATION: 9:00 AM - 10:30 AM

Instructor: Biplab Basak | Date: 20/09/2021

FULL MARKS: 35

QUESTION 1: Consider the statements: $M =$ "Mitali comes to the class", $D =$ "Diksha comes to the class", and $A =$ "Avani comes to the class".

Simran says "If Mitali or Diksha come to the class, then Avani comes too."
Ritika says "If Avani doesn't come to the class, then Mitali doesn't come either."
Era says "Diksha comes to the class provided Avani comes to the class."
Tushita says "At least one of Ritika and Era is correct."

(a) Formalize the statements of Simran, Ritika, Era, and Tushita into propositional logic.

(b) Let Simran's and Tushita's statements are represented by S and T respectively. Which of the following is/are true?

(i) S and T are equivalent.

(ii) $S \implies T$.

(iii) $T \implies S$.

Use a truth table to explain your reasoning. (2+4)

QUESTION 2: Let m be an even integer and n be an even natural number. By using 'a proof by contradiction' show that $p(\frac{m}{n}) \neq 0$, where $p(x) = x^3 + x + 1$. (6)

QUESTION 3: Prove by induction that for any integer $n > 23$, there exist nonnegative integers x and y such that $n = 7x + 5y$. (5)

QUESTION 4: Assume that the real line $\mathbb{R}$ has the least upper bound property. Consider the ordered set $X \subset \mathbb{R} \times \mathbb{R}$ with the dictionary order relation.

(a) Let $X = [-1, 1] \times [-1, 1]$ and A be the closed circular region in X given by $\{x \times y : x^2 + y^2 \leq \frac{1}{2}\}$ in the usual sense. Find the least upper bound of the set $A \subset X$ if X is the ordered set with the dictionary order relation.

(b) Let $X = [-1, 1] \times [-1, 1]$ and A be the open circular region in X given by $\{x \times y : x^2 + y^2 < 1\}$ in the usual sense. Find the least upper bound of the set $A \subset X$ if X is the ordered set with the dictionary order relation.

(c) Let $X = [-1, 1) \times [-1, 1]$ and $A = [-1, 1) \times (-1, 0)$. Find the least upper bound of the set $A \subset X$ if X is the ordered set with the dictionary order relation.

Give proper arguments for each case. (2+2+2)

QUESTION 5: Recall that a set $\mathbb{B} \subset \mathbb{R}$ is a basis for the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ if every element of $\mathbb{R}$ is written in a unique way as a finite linear combination of elements of $\mathbb{B}$, i.e., every element $v \in \mathbb{R}$ is **uniquely** expressed as

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$

where $c_1, c_2, \ldots, c_n \in \mathbb{Q}$, $v_1, v_2, \ldots, v_n \in \mathbb{B}$, and n is a natural number depending on v.

Given that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. Discuss whether the basis $\mathbb{B}$ is finite or countably infinite or uncountable. (6)

QUESTION 6: Find the cardinality of the following sets.

(a) Let $(S_n)_{n=1}^{\infty}$ be a sequence of sets such that the cardinality of S_n is same as $\mathbb{R}$, i.e., $\aleph_1$ for all $n \in \mathbb{N}$. Find the cardinality of $\cup_{n=1}^{\infty} S_n$.

(b) Let S be the set of all eventually zero sequences of real numbers, i.e., $S = \{(x_1, x_2, x_3, \ldots) : x_i \in \mathbb{R} \,\forall\, i \in \mathbb{N}$ and $a_i = 0$ where $i \geq m$ for some $m \in \mathbb{N}\}$. Find the cardinality of S.

Note that the natural number m is not fixed, it will vary with each sequence. Here, find $|S|$ and $| \cup_{n=1}^{\infty} S_n|$ in terms of $\aleph_0$ or $\aleph_1$ or $\aleph_2$ or $\cdots$. (3+3)