Department of Mathematics
MTL107:Numerical Methods and Computations

Major,

Max Marks: 50

Max Time: 2 Hours

Answer ALL questions

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1. (a) Given $g \in C[a,b]$, $g(x) \in [a,b]$ $\forall x \in [a,b]$ and $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, $\forall x \in (a,b)$. Then prove the existence and uniqueness of fixed point for the function $g(x)$ in $[a,b]$. [4 Marks]

(b) Apply the Gauss elimination with partial pivoting and 4−digit rounding arithmetic to the following linear system

$$0.003x_1 + 59.14x_2 = 59.17$$

$$5.291x_1 - 6.13x_2 = 46.78$$

and compare the results to the exact solution $x_1 = 10$ and $x_2 = 1$ [4 Marks]

2. (a)Using the given data, determine the coefficient of x^2 in $P(x)$, a polynomial of unknown degree and assume that all third-order forward differences are 1. [3 Marks]

x	0	1	2
$P(x)$	2	-1	4

(b) Using the given data, construct the least squares approximation of the form $y(x) = be^{ax}$. [5 Marks]

x_i	4.0	4.5	5.1	5.9	6.8
y_i	102	130	167	224	299

3. (a) Using Lagrange interpolation, derive a forward difference formula along with the error term for the first derivative $y'(x_0)$. [3 Marks]
(b) Using the above forward difference formula for $y'(x_0)$ derive an inequality for the total error $E(h)$. [2 Marks]
(c) Using the above inequality for $E(h)$, compute the optimal h to minimize $E(h)$ for the given initial value problem

$$y' = -y + 2, \quad 0 \leq x \leq 1, \quad y(0) = 0.$$

(Assume that the maximum round-off error is bounded by $\epsilon = 10^{-2}$). [3 Marks]

(Please Turn Over for the rest of the Questions)

4. Let $T(a,b)$ and $T(a,\frac{a+b}{2})+T(\frac{a+b}{2},b)$ be the single and double applications of the Trapezoidal rule respectively to approximate $\int_a^b f(x)dx$. Derive a relationship between [6 Marks]

$$\left|T(a,b)-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)\right|$$

and

$$\left|\int_a^b f(x)dx-T(a,\frac{a+b}{2})-T(\frac{a+b}{2},b)\right|.$$

5. (a) Suppose that $x_1, x_2, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \cdots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then prove that [6 Marks]

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

(b) Use Gaussian Quadrature with $n = 3$ in both the dimensions to approximate the following double integral: [6 Marks]

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5} \ln(x+2y) \quad dy \quad dx.$$

6. Compute the approximate solutions $y(0.5)$ and $y(1.0)$ using
   (a) Taylor's Series Method of order two
   and
   (b) a Runge-Kutta Method of order two,
   for the following initial value problem

$$y' = \cos x + e^x, \quad 0 \leq x \leq 1, \quad y(0) = 1$$

with $h = 0.5$.
Tabulate the computed approximate solutions along with the exact solutions and the corresponding absolute errors. [8 Marks]