INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
MTL 411 (FUNCTIONAL ANALYSIS)
Major Examination

DATE: 17/05/2021 Total Marks: 40 Time: 10:00 – 12:00 noon.

MARKS WILL BE AWARDED ONLY FOR THOSE ANSWERS WITH PROPER JUSTIFICATION

Question 1: If $\{x_n\}_n$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then show that

$$\lim_{n\to\infty} \langle x_n, y\rangle = 0,$$

for each $y \in \mathcal{H}$.

[4]

Question 2: Let $A \in \mathbb{C}$ and let $k : [0,1]\times[0,1] \to \mathbb{R}$, be defined by

$$k(x,y) = A\,\sin(x-y).$$

Show that if $|A| < 1$, then for any $f \in \mathcal{C}[0,1]$ there exists $g \in \mathcal{C}[0,1]$ such that

$$g(x) = f(x) + \int_0^1 k(x,y)\,g(y)\,\mathrm{d}y.$$

[6]

Question 3: Let X be an infinite dimensional Banach space and $P : X \to X$ be a linear map satisfying $P^2 = P$. Then prove that P is a bounded if and only if Image(P) and Ker(P) are closed subspaces of X.

[6]

Question 4: Define $T : l^2 \to l^2$ and $S : l^2 \to l^2$ by

$$T(x) = (0, x_1, x_2, \ldots)$$
$$S(x) = (x_2, x_3, \ldots),$$

where $x = (x_1, x_2, \ldots) \in l^2$. Show that T and S define continuous linear operator on l^2 and that $ST = I$ while $TS \neq I$, where I stands for identity operator.

[6]

Question 5: Let $X = \{u \in \mathcal{C}[0,1], u(0) = 0\}$ with its usual norm $||u||_X = \max_{t\in[0,1]}|u(t)|$. Consider the linear functional

$$T(u) = \int_0^1 u(t)\,\mathrm{d}t.$$

(a) Show that $T \in X^\star$ and compute $||T||$.

(b) Can one find some $u \in X$ such that $||u|| = 1$ and $T(u) = ||T||$? [4 + 2 = 6]

Question 6 : Let $X = \mathcal{C}[0,1]$ with its usual norm $||u||_X = \max_{t\in[0,1]}|u(t)|$. Let

$$C = \{u \in X : \int_0^1 |u(t)|^2 \, \mathrm{d}t < 1\}.$$

(a) Check that C is convex and symmetric and that $0 \in C$. Is C bounded in X?

(b) Compute the Minkowski functional p associated with C. [3 + 3= 6]

Question 7 : Assume that a sequence $\{x_n\}_n$ in an inner product space $\mathcal{H}$ satisfies $\langle x_n, x\rangle \to ||x||^2$ and $||x_n|| \to ||x||$. Show that

$$x_n \to x \quad \text{in } \mathcal{H}.$$

[6]

—–ALL THE BEST—–