$\|T(x)\| \le M\|x\|$

MTL 411 – FUNCTIONAL ANALYSIS – MINOR 1

DEPT. OF MATHEMATICS, IIT DELHI
MAX MARKS - 25

1. Answer the following [10 Marks(4 + 6)]

(1) Let V be a vector space and suppose that $\| \|_1$ and $\| \|_2$ are norms on V whose corresponding topologies are T_1 and T_2. Show that if V is complete with respect to both norms and if $T_1 \supset T_2$, then $T_1 = T_2$.

(2) Give an example of:

(a) A normed linear space that is a Banach Space and,

(b) A normed linear space that is NOT a Banach Space.

Justify your assertions.

2. Answer the following [8 Marks(2+3+3)]

Let $(X_i, \| . \|_i), i = 1, 2, \dots n$ be real normed spaces, and $X = X_1 \times X_2 \times \dots \times X_n$.

(a) Define $\| . \|: X \to \mathbb{R}^+$ as

$$\| x \| = \max_{i=1,2,\dots n} \| x_i \|_i, x = (x_1, x_2, \dots, x_n) \in X$$

Prove that $\| . \|$ defines a norm on X.

(b) Let $\phi_{ik} : X_k \to X_i$ be linear operators. Define $\phi : X \to X$ as

$$(\phi x)_i = \Sigma_{k=1}^{n} \phi_{ik} x_k, i = 1, 2, \dots n$$

Prove that ϕ is bounded if and only if each ϕ_{ik} is bounded.

3. Answer the following [7 Marks(5+2)]

(1) Prove that the unit closed ball $B(X)$ is compact for any finite dimensional normed linear space X.

(2) Hence prove that any linear functional on a finite dimensional normed linear space X is bounded.

$(\phi x)_i = \sum_{k=1}^{n} \phi_{ik} x_k \le M$

$\phi(x_1, x_2, \dots x_n) \le M$

$X, \|\cdot\|$

$\|T(x)\| \le M\|x\|$