MTL411: Functional Analysis Prof. Sreenadh

Minor-1

Max. Marks: 20

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1. Prove that if $x \in l^q$, for some $q \geq 1$, then $x \in l^p$, for all $p > q$ and for any $x \in l^q$, [5]

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p.$$

2. Let $X, Y$ be normed linear spaces and let $T : X \to Y$ be a linear operator. Then show that there exists $A > 0$, such that

$$\|Tx\| \geq A\|x\|, \ \ \forall x \in X$$

if and only if $T$ is 1-1 and $T^{-1} : Range(T) \to X$ is continuous. [5]

3. Consider the real vector space $C^2([0,1])$ of functions that are twice differentiable in $(0,1)$ and $f''$ is continuous up to the end points of the interval $[0,1]$. Then

   (a) Show that $C^2([\overset{0}{\cancel{a}}, \overset{1}{\cancel{b}}])$ is a Banach space with respect to the norm

   $$\|f\|_2 = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$$

   where $\|f\|_\infty = \max_{[0,1]} |f(x)|$.

   (b) Show that (using closed graph theorem)

   $$\|f\|_1 = \|f\|_\infty + \|f''\|_\infty$$

   is an equivalent norm. [2+3]

4. Let $T : X \to X$ be a bounded linear operator on a Banach space $X$ such that $\|T\| < 1$. Then show that

   (a) $\sum_{n=0}^{\infty} T^n$ is a bounded linear operator from $X$ to $X$

   (b) $I - T$ is bijective. [3+2]