Department of Mathematics, IIT Delhi

MTL725: Major Examination

Time: 2 hour **Date:** 04-05-2019 **Total Marks:** 40

Q.1) Let $N(t)$ be a Poisson process with intensity $\lambda > 0$.

a) Show that, for a pre-assigned $\epsilon > 0$, $\mathbb{P}\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \epsilon\right) \to 0$ as $t \to \infty$.

b) Prove that $X(t) = \exp\{-\theta N(t) + \lambda t(1 - e^{-\theta})\}$, $\theta \in \mathbb{R}$ is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(N(s) : 0 \leq s \leq t)$.

3+4 marks

Q.2) Let $\{N(t) : t \geq 0\}$ be a renewal process with i.i.d. inter-arrival times $\{T_i\}$ such that $\mathbb{E}[T_i] = \mu$ and $\mathrm{Var}[T_i] = \sigma^2$.

i) State the central limit theorem for $\{N(t) : t \geq 0\}$.

ii) Suppose $\{T_i\}$ has a density function $f(x)$ given by

$$f(x) = \theta - \frac{x\theta^2}{2}, \quad 0 \leq x \leq \frac{2}{\theta} \quad \text{for some } \theta > 0.$$

Show that $\dfrac{\mathbb{E}(N(t))}{t} \to \dfrac{3\theta}{2}$ as $t \to \infty$.

1+4 marks

Q.3) Let $\{N(t) : t \geq 0\}$ be a renewal process with i.i.d. inter-arrival times $\{T_i\}$ such that $\mathbb{E}[T_i^2] < \infty$. Let $\delta_t := t - S_{N(t)}$ be the spent lifetime at time t.

i) Show that $\displaystyle\int_0^{S_{N(t)}} \delta_s\, ds = \sum_{i=1}^{N(t)} \frac{T_i^2}{2}$.

ii) Let $R(t) = \displaystyle\sum_{i=1}^{N(t)} R_i$, where $R_i = \frac{T_i^2}{2}$. Show that $R(t) \leq \displaystyle\int_0^t \delta_s\, ds < R(t) + R_{N(t)+1}$.

3+2 marks

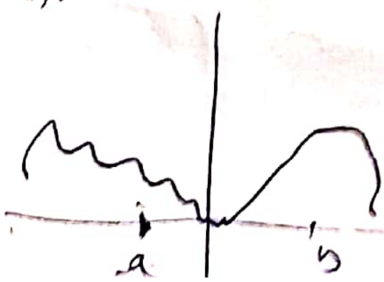
Q.4) Let $\{B_t\}$ be a standard Brownian motion. Define the stopping time

$$\tau = \inf\{t : B_t = -a \text{ or } B_t = b\} \quad (a, b > 0).$$

Let $p = \mathbb{P}(B_\tau = b)$ and $1 - p = \mathbb{P}(B_\tau = -a)$.

a) Show that $p = \frac{a}{a+b}$.

b) Show that the expected value of τ is ab.



a. = a

2+3 marks

Q.5) Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain with state space $S = \{0, 1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

a) How many classes are there?

b) Classify the states of the Markov chain as positive recurrent, null recurrent or transient.

c) Calculate f_{34} and f_{02}.

1+4+1 marks

Q.6) Let $\{X(t) : t \geq 0\}$ be a linear growth birth and death process with birth rate resp. death rate λ_n resp. μ_n given by

$$\lambda_n = n\lambda + a; \quad \mu_n = n\mu, \quad \lambda, \mu, a > 0.$$

Show that the stationary distribution $\pi = (\pi_i)_{i \in \mathbb{N}}$ satisfies the formula: for $\lambda < \mu$,

$$\pi_i = \frac{\left(\frac{\lambda}{\mu}\right)^i \prod_{k=0}^{i-1}\left(k + \frac{a}{\lambda}\right)}{i!}\left(1 - \frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}}.$$

5 marks

Q.7) Consider a two-state continuous-time Markov chain that spends an exponential time with rate λ in state 0 before going to state 1, where it spends an exponential time with rate μ before returning to state 0.

i) Write down its rate matrix Q.

ii) Show that, $p_{00}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$.

iii) Verify that, $\lim_{t \to \infty} p_{11}(t) = \frac{\lambda}{\lambda + \mu}$.

1+3+3 marks

Best of Luck!!!