Department of Mathematics, IIT Delhi

MTL733: Major Exam.

Time: 1 hour 50 minutes **Date:** 17-11-2021 **Total Marks:** 45

Q.1) Let $B(\cdot)$ be a one-dimensional Brownian motion.

i) Consider a stopping time $\tau := \inf\{t \geq 0 : B(t) = 5\}$. Explain whether $\mathbb{E}[\tau^2]$ is finite or not. What can you say about $\mathbb{E}[\tau^k]$ for $k \geq 3$.

ii) Consider the following non-negative stochastic process $X(t)$:

$$dX(t) = (1 - 2X(t))\,dt + 3\sqrt{X(t)}\,dB(t), \quad X(0) = 3.$$

a) Find the mean of $X(t)$.

b) Calculate $\mathbb{E}[Y^2]$ where $Y(t) := e^{2t}X(t)$.

c) Derive the second moment of $X(t)$.

(2+1)+(3+3+1) marks

Q.2) Let $B(\cdot)$ be a one dimensional Brownian motion. Consider the stopping time

$$\tau = \inf\{t : X(t) = 3\}, \quad X(t) = 5t + B(t).$$

a) Show that there exist a probability measure Q and a Brownian motion $\tilde{B}(\cdot)$ such that

$$\mathbb{P}\big(\tau \in dt\big) = \mathbb{E}_Q\Big[\exp\{5\tilde{B}(t) - \frac{25}{2}t\}\mathbf{1}_{\{\tau \in dt\}}\Big].$$

where $\mathbb{P}\big(\tau \in dt\big)$ denotes the probability of τ being in infinitesimal interval.

b) Deduce the probability density function (pdf) of τ.

c) Find the Ito-representation form for the martingale

$$M(t) := \mathbb{E}\big[B^2(T)\big|\mathcal{F}_t\big],\ 0 \leq t \leq T,$$

where $\mathcal{F}_t$ is the natural filtration generated by Brownian motion $B(\cdot)$.

5+2+3 marks

Q.3) Let $B(\cdot)$ be a one-dimensional Brownian motion and the filtration $\{\mathcal{F}_t\}$ is generated by Brownian motion only. Consider a market $X(t) = \big(S_0(t), S(t)\big)$ given by

$$dS(t) = tS(t)\,dt + S(t)\,dB(t), \quad S(0) = 2$$
$$dS_0(t) = 3S_0(t)\,dt, \quad S_0(0) = 1\,.$$

a) Show that there exists a risk-neutral measure Q on $\mathcal{F}_T$ for the market $X(t)$.

b) Let M be a martingale under Q. Show that there exist a Q-Brownian motion $\bar{B}(\cdot)$ and a adapted process $\bar{f}(t)$ such that

$$M(t) = \mathbb{E}[M(0)] + \int_0^t \bar{f}(t)\, d\bar{B}(t), \quad 0 \le t \le T.$$

5+7 marks

Q.4) Consider a stock whose differential is

$$dS(t) = tS(t)\, dt + t^2 S(t)\, d\tilde{B}(t)$$

where $\tilde{B}(\cdot)$ is a Brownian motion under risk-neutral measure. Let $T > 0$ be given.

a) Show that $S(T)$ is of the form $S(0)e^X$ where $X \sim \mathcal{N}\big(\frac{T^2}{10}(5 - T^3), \frac{T^5}{5}\big)$.

b) Consider a call option whose value at time 0 is

$$C(0, S(0)) = \mathbb{E}_Q\Big[\exp\{-\int_0^T s\, ds\}\big(S(T) - K\big)^+\Big].$$

Let

$$\mathrm{BSM}\big(T, x; K, r, \sigma\big) := xN(d_1) - Ke^{-rT}N(d_2)$$

denotes the value of a European call option expiring at time $T$with strike price K when the underlying stock has constant volatility σ and interest rate r, where

$$d_2 := \frac{\log(\frac{x}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_1 = d_2 + \sigma\sqrt{T},$$

and $N(\cdot)$ is the cumulative standard normal distribution. Show that

$$C(0, S(0)) = \mathrm{BSM}\Big(T, S(0); K, \frac{T}{2}, \frac{T^2}{\sqrt{5}}\Big).$$

c) Suppose that a stock (with constant volatility and interest rate) sells today for 100, the value of the call option is 6, the value of the put option is 5 and both options have the same strike price, 100, with one year expiry time. What is the risk-free interest rate?

5+6+2 marks

Best of Luck!!!