Department of Mathematics, IIT Delhi

MTL733: Minor Exam.

Time: 1 hour 30 minutes **Date:** 23-09-2021 **Total Marks:** 40

Q.1) Let $B(\cdot)$ be a one-dimensional Brownian motion. Let $X(t)$ be a stochastic process which satisfies the SDE

$$dX(t) = -\frac{X(t)}{1-t}\,dt + dB(t), \quad 0 \le t < 1$$
$$X(0) = 0.$$

i) Show that $X(t)$ is given by the formula:

$$X(t) = (1-t)\int_0^t \frac{1}{1-s}\,dB(s), \quad 0 \le t < 1.$$

ii) Explain whether the process $X(t)$ is normally distributed or not.

iii) Find the limit: $\lim_{t\to 1-} \mathbb{E}[X^2(t)]$.

(6+2+3) marks

Q.2) Let $\mathbf{B}(t) = \big(B_1(t), B_2(t), \ldots, B_m(t)\big)$ be a m-dimensional Brownian motion. For $c, \alpha_1, \alpha_2, \ldots, \alpha_m$ constants, define a stochastic process $X(t) := \exp\Big\{ct + \sum_{j=1}^m \alpha_j B_j(t)\Big\}$.

i) Show that

$$dX(t) = \Big(c + \frac{1}{2}\sum_{j=1}^m \alpha_j^2\Big)X(t)\,dt + X(t)\Big(\sum_{j=1}^m \alpha_j\,dB_j(t)\Big).$$

ii) Let $e(t) := \mathbb{E}[X(t)]$. Find condition on c (in terms of α_j's) such that $\lim_{t\to\infty} e(t) = 0$.

iii) Show that $\mathbf{X}(t) := \boldsymbol{U}\mathbf{B}(t)$ is a 2-dimensional Brownian motion, where $\boldsymbol{U} = \begin{pmatrix} \cos(x_0) & \sin(x_0) \\ -\sin(x_0) & \cos(x_0) \end{pmatrix} \in M^{2\times 2}$ and $\mathbf{B}(t)$ is a 2-dimensional Brownian motion.

(3+4)+3 marks

Q.3) Let $B(\cdot)$ be a one-dimensional Brownian motion. Consider a stock price and state price density processes $Y(t)$ and $\xi(t)$ respectively

$$dY(t) = \alpha Y(t)\,dt + \sigma Y(t)\,dB(t); \quad \xi(t) = \exp\{-\theta B(t) - (r + \frac{1}{2}\theta^2)t\},$$

where α, σ and r are constants and $\theta = \frac{\alpha - r}{\sigma}$.

a) Show that

$$d\xi(t) = -\theta\xi(t)\,dB(t) - r\xi(t)\,dt.$$

b) Consider the stochastic process

$$dX(t) = rX(t)\,dt + \gamma(t)(\alpha - r)Y(t)\,dt + \sigma\gamma(t)Y(t)\,dB(t),$$

where $\gamma(t)$ is a given adapted process. Show that $Z(t) := \xi(t)X(t)$ is a martingale.

c) Suppose $f, g \in \mathcal{Y}(0,T)$ and that there exist constants A_1, A_2 such that $\mathbb{P}$-a.s.,

$$A_1 + \int_0^T f(t)\,dB(t) = A_2 + \int_0^T g(t)\,dB(t).$$

Show that $A_1 = A_2$ and $f(t,\omega) = g(t,\omega)$ for a.a. $(t,\omega) \in [0,T] \times \Omega$.

3+4 +4 marks

Q.4) Let $\mathbf{B}(t) = \big(B_1(t), B_2(t)\big)$ be a 2-dimensional Brownian motion.

a) Check whether the process $X(t) := B_1(t)B_2(t)$ is a martingale or not.

b) Explain whether the processes $M_1(t)$ and $M_2(t)$ defined by

$$M_1(t) = \frac{1}{\sqrt{5}}B_1(t) + \frac{2}{\sqrt{5}}B_2(t); \quad M_2(t) = \frac{3}{5}B_1(t) + \frac{4}{5}B_2(t)$$

are one-dimensional Brownian motion or not.

c) Is $\mathbf{M}(t) = \big(M_1(t), M_2(t)\big)$ a 2-dimensional Brownian motion? Justify your answer.

4+2+2 marks

Best of Luck!!!