DEPARTMENT OF MATHEMATICS
MTL 741: Fractal Geometry

Major *Marks - 45*

All questions are 5 marks each. Maximum marks will be 45.

(1.) Let X be a topological space and let $n \in \mathbb{N}$. TFAE

(1) $dim(X) \leq n$.

(2) for every $\epsilon > 0$, there exists a finite open cover α of X, such that $mesh(\alpha) \leq \epsilon$ and $ord(\beta) \leq n$.

(3) there exists a sequence of finite open cover $\{\alpha_k\}_{k \in \mathbb{N}}$ of X, such that $\lim\limits_{k \to \infty} mesh(\alpha_k) = 0$ and that $ord(\alpha_k) \leq n$ for all $k \in \mathbb{N}$.

(2.) Let $f : I\!R \to I\!R$ be the function $f(x) = x^2$. For any subset $F \subset I\!R$ show that $dim_H(F) = dim_H(f(F))$.

(3.) Let $X = \{0,1\}^{\mathbb{N}}$ and define a metric on X as $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$.
Define $F : X \to X$ as $F(x) = F(x_1x_2x_3\ldots) = 1x_1x_2x_3\ldots$.
Show that F is a contraction mapping on X and find its fixed point.

(4.) Let $\{X, S_1, \ldots, S_n\}$ be an IFS with contractivity factor s where (X, d) is a complete metric space. Let A be its attractor.
Let (Σ, ρ) be the associated code space.
For each $\alpha \in \Sigma$, $N \in \mathbb{N}$ and $x \in X$ define $\phi(\alpha, N, x) = S_{\alpha_1} \circ \ldots \circ S_{\alpha_N}(x)$, and

$$\phi(\alpha) = \lim_{N \to \infty} \phi(\alpha, N, x)$$

Show that $\phi : \Sigma \to A$ is continuous and onto.

(5.) Consider the IFS

$$\{[0,1], f_1(x) = \frac{x}{3}, f_2(x) = \frac{x}{4} + \frac{1}{2}\}.$$

Find the contractivity factor of this IFS. Show that its attractor is perfect and totally disconnected.

(6.) Show that any repelling periodic point of $z^2 + c$ will be on the Julia set J_c corresponding to the constant c.

(7.) Describe the difference between the concepts of the basin of attraction and the prisoner set. Give an example to illustrate your assertions.

(8.) What is meant by a conjugate map? Prove that any quadratic polynomial is conjugate to the map $z^2 + c$ for some $c \in \mathbb{C}$.
Describe the Julia set of $p(z) = z^2 + 2z$ by finding its conjugate.

(9.) Let $f(z) = z^2 + c$ and $A(\infty)$ denote the basin of attraction of ∞ for f and $J(f)$ denotes the filled in Julia set for f. Prove that:

a. $f(A(\infty)) \subseteq A(\infty)$ and $f(J(f)) \subseteq J(f)$.

b. $A(\infty)$ and $J(f)$ are disjoint and $A(\infty) \cup J(f) = \mathbb{C}$.