Max Marks: 15 | Max Time: 1 hr

1. Let $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathbb{N}_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \right)$.

   (a) Determine the constant-density ellipse that contains data from this distribution with probability equals 0.90.

   (b) Determine the direction ratios and lengths of the major and minor axes of the ellipse obtained in part (a). Provide a (rough) sketch of this ellipse. [4]

2. A company makes a candy bar in different three sizes: small (X_1), regular (X_2), and big (X_3). The joint distribution of the weight (in gms) of the candy bars $\underset{\sim}{X} = (X_1, X_2, X_3)^T$ follow a multivariate normal distribution with parameters $\underset{\sim}{\mu} = (3, 5, 7)^T$ and $\Sigma = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix}$.

   (a) What is the probability that the weight of a regular bar is greater than 8 gm, given that the small size bar weighs 2 gm and the big size bar weighs 10 gm?

   (b) Determine $P(X_1 - 2X_2 + X_3 < 5)$. [4]

3. Let A be a $p \times p$ symmetric positive definite matrix. Prove that, for all symmetric positive definite matrices M,

$$\frac{1}{|M|^2} \exp\left(-\frac{1}{2}\text{trace}(M^{-1}A)\right) \leq \frac{(16)^p}{|A|^2} \exp(-2p).$$ [4]

4. Let $\underset{\sim}{X} = (X_1, X_2, X_3)^T \sim \mathbb{N}_3(\underset{\sim}{\mu}, \Sigma)$, $\Sigma = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix}$, and $\underset{\sim}{a} = (1, -1, 1)^T$. Determine the value of the vector $\underset{\sim}{r} = (r_1, r_2, r_3)^T$, where r_i denotes the correlation between X_i and $\underset{\sim}{a}^T \underset{\sim}{X}$. [3]