DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
MAJOR TEST 2021-2022 FIRST SEMESTER
MTL 781 (FINITE ELEMENT ANALYSIS AND APPLICATIONS)

Time: 1 hour 40 Minutes **Max. Marks: 40**

$**$ **Answer to each question should begin on a new page.** $**$
$**$ **Complete and precise answers deserve full credit.** $**$
$**$ **All parts of a question must be answered at one place.** $**$
$**$ **All notations are standard. Exhibit clearly all the steps.** $**$

1a. Compare Fourier Series method with Galerkin method. (2)

1b. Solve by the method of least squares, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 10xy \quad \text{in} \quad \Omega,$$

where Ω is a square bounded by $x = 0, x = 1$ and $y = 0, y = 1$. It is given that $u = 2$ on the boundary. Take trial solution (approximate solution) $\widetilde{U} = 2 + axy(1 - x)(1 - y)$. (7)

2. Let Ω be a bound open set. Show that the bilinear form defined by

$$a(u, v) = \int_\Omega \Delta u \; \Delta v \; dx, \quad u, v \in H_0^2(\Omega),$$

where Δ is the Laplacian operator, is symmetric, continuous and $H_0^2(\Omega)-$elliptic. (1+3+2=6)

3a. Prove that the Heaviside function on R defined by

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

has no weak derivative. (3)

3b. Let $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ be a bilinear form defined by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv,$$

where Ω is a bounded subset of R^n. Let $f \in L^2(\Omega)$. Show that there exists a quadratic functional $J : H_0^1(\Omega) \to R$ of the form

$$J(v) = \frac{1}{2} a(v, v) - \int_\Omega fv$$

such that if $u \in H_0^1(\Omega)$ is a solution of $a(u, v) = \int_\Omega fv, \;\; \forall \; v \; \in H_0^1(\Omega)$, then $J(u) \leq J(v) \;\; \forall \; v \; \in H_0^1(\Omega)$. (4)

4a. Derive quadratic interpolation polynomial in areal coordinates (3)

4b. Obtain linear interpolation polynomial in volume coordinates. (2)

5. Derive integration formula for

$$\int\int_R L_1^{\alpha} L_2^{\beta} L_3^{\gamma} \; dA$$

where R is the triangular region; dA is elemental area of the triangular element; L_1, L_2, L_3 are areal coordinates; α, β, γ are positive constants.
Hence find the value of

$$\int\int_R L_1 L_2^2 L_3^3 \; dA$$

(6)

6. Consider the poisson equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \cos \pi x \text{ in } \Omega \subset R^2$$

whereΩ is the unit square. The origin of the coordinate system is taken at the left corner of the square. Consider the Dirichlet boundary conditions

$$u = 0, \text{ on } \Gamma.$$

Obtain N-parameter Ritz solution. (7)