Indian Institute of Technology Delhi
Minor-Exam
MTL-860 (Linear Algebra)

Time: 1 Hour (+15 minutes uploading time) Max. Marks: 25

Attempt all questions. All notations are standard. All parts of a question must be answered at one place. Exhibit clearly all the steps. No query will be entertained. ***In each question write all the steps/calculations clearly. No marks will be given for direct answers.***

Note: "Homomorphism" and "Linear Transformation" are equivalent words.

1. (a) Let V be an inner product space. For $u,\ v \in V$ such that $||u|| = 5$ and $||v|| = 3$ then $|\langle u, v\rangle| = 16$. (True/False, Justify your answer) [1 mark]

   (b) Let V be a finite dimensional vector space. Suppose W_1, W_2, W_3 are subspace of V such that $W_1 \cap W_2 = \{0\} = W_2 \cap W_3 = W_3 \cap W_1$ then $dim(W_1 + W_2 + W_3) = dim(W_1) + dim(W_2) + dim(W_3)$. (True/False, Justify your answer) [2 mark]

   (c) Find two linear transformation T_1 and T_2 on $\mathbb{R}^2$ over $\mathbb{R}$ such that $T_1 \circ T_2 = 0$ (zero liner transformation) but $T_2 \circ T_1 \neq 0$. '$\circ$' denotes composition of function. [1 marks]

   (d) Prove/disprove the following statement [1 marks]
   If V is a vector space over a filed $\mathbb{F}$ and $v_1, v_2, v_3 \in V$ are vectors such that

      i. $v_1 \neq 0$;
      ii. $v_2 \notin span(v_1)$;
      iii. $v_3 \notin span(v_1, v_2)$;

   then v_1, v_2 and v_3 are linearly independent.

2. Let V_1 and V_2 be two subspaces of $\mathbb{R}^4$ over $\mathbb{R}$ given by [3 marks]

$$V_1 = \{(a, b, c, d) \in \mathbb{R}^4 | b - 2c + d = 0\}$$
$$V_2 = \{(a, b, c, d) \in \mathbb{R}^4 | a = d, b = 2c\}$$

   find the basis and dimension of $(i) V_1$ $(ii) V_2$ $(iii) V_1 + V_2$

3. (a) Which of the following is inner product? Justify . [1 marks]

      i. $\langle(x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + y_1y_2$ on $\mathbb{C}^2$ over $\mathbb{C}$
      ii. $\langle(x_1, y_1), (x_2, y_2)\rangle = x_1y_1 + x_2y_2$ on $\mathbb{R}^2$ over $\mathbb{R}$

   (b) There exists an inner product on $\mathbb{R}^2$ over $\mathbb{R}$ such that $\langle(1, 0), (0, 1)\rangle = 2$. (True/false justify your answer) [3 marks]

   (c) Find the best approximation of $v = (1, 2, 3) \in \mathbb{R}^3$ over $\mathbb{R}$ by the vectors of subspace $W = span\{(1, 1, 0)\}$ with inner product $\langle(x_1, y_1, z_1), (x_2, y_2, z_2)\rangle = x_1y_1 + x_2y_2 + z_1z_2$. [3 marks]

4. (a) If V is a finite dimensional vector space and $v_1 \neq v_2$ in V then prove that there is an $f \in \hat{V}$ (dual of V) such that $f(v_1) \neq f(v_2)$. [2 marks]

(b) Apply Gram-Schmidt orthogonalization process to find an orthonormal basis for vector space $\mathbb{P}_2(x)$, set of polynomial of degree at most 2 with for basis $B = \{1, x, x^2\}$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$$

[3 marks]

5. (a) Find the subspace W of $V = \mathbb{R}^4$ over $\mathbb{R}$ such that annihilator of W, $Ann(W) = span\{f_1, f_2, f_3\}$ where [2 marks]

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) =& x_1 + x_2 + 2x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) =& 2x_2 + x_4 \\ f_3(x_1, x_2, x_3, x_4) =& -2x_1 - 4x_3 + 3x_4 \end{aligned}$$

(b) If W_1 and W_2 are subspaces of V, which is finite-dimensional, describe $Ann(W_1 + W_2)$ in terms of $Ann(W_1)$ and $Ann(W_2)$. [3 marks]