Major: Part B
02-05-2019

PYL112: Quantum Mechanics

II Semester 2018-19

Answer any FIVE questions

Time: 1.5 hours
Max. Marks: 30

NOTE: All symbols used have their usual meaning. Each question carries 6 **marks**. Some useful mathematical expressions are given at the end of the questions.

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1) Calculate from first principles the Clebsch-Gordan coefficients for $j_1 = 1$ and $j_2 = \frac{1}{2}$. Use the calculated coefficients to write the wave functions for the p-states of the electron $\left(l = 1, s = \frac{1}{2}\right)$ for $j = \frac{3}{2}$ $\left(m_j = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ and $j = \frac{1}{2}$ $\left(m_j = \frac{1}{2}, -\frac{1}{2}\right)$.

[3+3 = 6 Marks]

2) Consider the radial wave function $R_{nl}(r)$ of a hydrogen atom and its corresponding radial probability density $P_{nl}(r)$. Calculate the following:

a) The expectation value $\langle r \rangle_{21}$ for the hydrogen atom. Also find the value of r at which the radial probability density reaches its maximum for the state $n = 2$, $l = 1$. Compare these two values and comment (giving a proper reason) on their difference if any.

b) The width of the probability density distribution for r w.r.t. the above radial wavefunction.

[3+2 = 5 Marks]

3) The wave function of an electron in a hydrogen atom is given by

$$\left|\psi_{21m_lm_s}(r,\theta,\varphi)\right\rangle = R_{21}(r)\left[\sqrt{\frac{2}{5}}Y_{10}(\theta,\varphi)\left|\frac{1}{2},+\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}Y_{11}(\theta,\varphi)\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right],$$

Y_{10}

Where $\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle$ are the spin state vectors.

a) Calculate $\hat{J}_z\left|\psi_{21m_lm_s}(r,\theta,\varphi)\right\rangle$, i.e., the *z-component* of the electron's *total* angular momentum.

b) If you measure $\hat{\vec{J}}^2$, what values will you obtain? What are the corresponding probabilities?

c) If you measure the *z-component* of the electron's *orbital* angular momentum, what values will you obtain? What are the corresponding probabilities?

d) Calculate $\left\langle \psi_{21m_lm_s}\right|\hat{L}_-\left|\psi_{21m_lm_s}\right\rangle$ and $\left\langle \psi_{21m_lm_s}\right|\hat{S}_z\left|\psi_{21m_lm_s}\right\rangle$.

[1+3+1+2 = 7 Marks]

$\frac{3}{2} \times \frac{5}{2} + \frac{1}{4}$

$\frac{15+1}{4}$ 4

4) Consider the electron's spin operator: $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$, where the components of $\hat{\vec{\sigma}}$ are the Pauli spin matrices.

a) Find the eigenvalues and eigenstates of $\hat{\vec{S}}$ in the direction of a unit vector $\vec{n}$; assume $\vec{n}$ lies in the xz plane.

b) Find the probability of measuring $\hat{S}_z = +\frac{\hbar}{2}$.

[3+2 = 5 Marks]

$1 + ($ $(1+\sqrt{2})(1-\sqrt{2})$ $(\sqrt{2}-1)$

5) Consider a system under translational symmetry.
(i) Use an appropriate form (stating proper reasons) of the operator $\hat{T}(\varepsilon)$ for an *infinitesimal* translation by ε and show that momentum is the generator of such translations. Give proper reasons in each step.
(ii) What does translational *invariance* imply? Show that momentum for the system is conserved under translational *invariance*.
(iii) What should be the form of the operator $\hat{T}(a)$ for a finite translation by 'a'? Deduce it from part (i) above.
[3+2+2 = 7 Marks]

Table 1: Spherical Harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z)=\frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z)=\sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z)=\mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z)=\sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z)=\mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z)=\mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

Table 2: First few radial wave functions $R_{nl}(r)$ of the hydrogen atom

$R_{10}(r)=2a_0^{-3/2}e^{-r/a_0}$	$R_{21}(r)=\frac{1}{\sqrt{6a_0^3}}\frac{r}{2a_0}e^{-r/2a_0}$
$R_{20}(r)=\frac{1}{\sqrt{2a_0^3}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}$	$R_{31}(r)=\frac{8}{9\sqrt{6a_0^3}}\left(1-\frac{r}{6a_0}\right)\left(\frac{r}{3a_0}\right)e^{-r/3a_0}$
$R_{30}(r)=\frac{2}{3\sqrt{3a_0^3}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$	$R_{32}(r)=\frac{4}{9\sqrt{30a_0^3}}\left(\frac{r}{3a_0}\right)^2 e^{-r/3a_0}$

Table 3: First few Legendre polynomials and associated Legendre functions.

Legendre Polynomials	Associated Legendre Functions
$P_0(\cos\theta) = 1$	$P_1^1(\cos\theta) = \sin\theta$
$P_1(\cos\theta) = \cos\theta$	$P_2^1(\cos\theta) = 3\cos\theta\sin\theta$
$P_2(\cos\theta) = (3\cos^2\theta - 1)/2$	$P_2^2(\cos\theta) = 3\sin^2\theta$
$P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$	$P_3^1(\cos\theta) = 3\sin\theta(5\cos^3\theta - 1)/2$
$P_4(\cos\theta) = (35\cos^4\theta - 30\cos^2\theta + 3)/8$	$P_3^2(\cos\theta) = 15\sin^2\theta\cos\theta$
$P_5(\cos\theta) = (63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)/8$	$P_3^3(\cos\theta) = 15\sin^3\theta$

Table 4: First few Laguerre polynomials and associated Laguerre polynomials.

Laguerre polynomials $L_k(r)$	Associated Laguerre polynomials $L_k^N(r)$
$L_0 = 1$	
$L_1 = 1 - r$	$L_1^1 = -1$
$L_2 = 2 - 4r + r^2$	$L_2^1 = -4 + 2r$, $L_2^2 = 2$
$L_3 = 6 - 18r + 9r^2 - r^3$	$L_3^1 = -18 + 18r - 3r^2$, $L_3^2 = 18 - 6r$, $L_3^3 = -6$
$L_4 = 24 - 96r + 72r^2 - 16r^3 + r^4$	$L_4^1 = -96 + 144r - 48r^2 + 4r^3$, $L_4^2 = 144 - 96r + 12r^2$, $L_4^3 = 24r - 96$, $L_4^4 = 24$
$L_5 = 120 - 600r + 600r^2 - 200r^3 + 25r^4 - r^5$	$L_5^1 = -600 + 1200r - 600r^2 + 100r^3 - 5r^4$, $L_5^2 = 1200 - 1200r + 300r^2 - 20r^3$, $L_5^3 = -1200 + 600r - 60r^2$, $L_5^4 = 600 - 120r$, $L_5^5 = -120$