

Midterm Exam (PYL101)

Date: 11/01/2022

Time: 10 AM - 12 NOON (2 Hrs)

Q 1. (a) Consider a vector function $\mathbf{f} = \frac{\hat{r}}{r^2}$. Sketch this function and calculate its divergence. Explain your answer on the divergence. (3 marks)

(b) Using the properties of Dirac delta function, evaluate the integral

$$\int_0^3 (x^3 + 2x + 1)(x + 2) dx \quad (3 \text{ marks})$$

Q 2. For a thick spherical shell having no free charges, the inner and the outer radii are R_1 and R_2 , respectively. The shell is made of a dielectric material with frozen-in polarization $(\mathbf{r}) = \frac{2k}{r} \hat{r}$. By calculating the bound charges, find the electric field \mathbf{E} in the regions: i) $r < R_1$ and ii) $R_1 < r < R_2$ (6 marks)

Q 3. Steady current I is flowing through a long cylinder of radius R . If the current I is distributed (in the cylinder) in such a way that volume current density J is proportional to the distance from the axis of the cylinder, calculate magnetic field \mathbf{B} for: i) inside the cylinder, ii) outside the cylinder. (6 marks)

Q 4. A coaxial cable consists of two long cylindrical tubes with their radii R_1 and R_2 , respectively. These two tubes are separated by an insulating material with magnetic susceptibility χ_m . A uniformly distributed (over the surface) current I flows down the inner tube and returns along the outer tube. By calculating the magnetization and bound currents in the system, find the magnetic field \mathbf{B} in the region between the tube. (7 marks)

Q 5. Write mathematical expressions for the vector electric and magnetic field components of a plane electromagnetic wave of wavelength 800 nm travelling along a direction (3,2,1) from the origin and polarized in the xy-plane. What would be results for a dielectric medium with dielectric constant 3.2. (5 + 2 marks)

Q 6. A plane monochromatic wave of wavelength 800 nm hits still surface of sea water from air at normal. If the conductivity of the sea water is 2×10^{-2} Siemens and refractive index is 5, what would be the electric and magnetic fields measured by an observer sitting in a ship 10 km beneath the surface. (Given, $\mu = \mu_0$ and $\epsilon_0 = 8.86 \times 10^{-12} \text{ C}^2/\text{N.m}^2$) (6 marks)

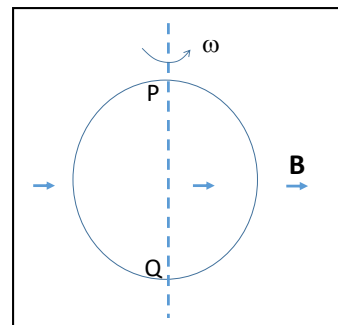
Q 7. Consider the device based on force transducer as discussed in the class for the measurement of radiation pressure. Using same device but without actually measuring the force in the transducer, think and discuss an alternate all-optical way for the measurement of radiation pressure by using wave packets instead. (3 marks)

Q 8. A thin metal ring of radius R is rotating with angular speed ω about an axis QP in a uniform magnetic field \mathbf{B} as shown in the figure. Calculate the EMF and the potential difference V_{PQ} for the following cases. Show all the steps.

(i) The uniform magnetic field fills the entire space where the ring is placed. (4 marks)

(ii) The magnetic field is like (i) but now directed along QP . (2 marks)

(iii) The magnetic field is like in (i) but now fills the entire right half space (on the right side of PQ) only. (3 marks)



VECTOR IDENTITIES

Triple Products

$$(1) \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$(2) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Product Rules

$$(3) \quad \nabla(fg) = f(\nabla g) + g(\nabla f)$$

$$(4) \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$(5) \quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$(6) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$(7) \quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$(8) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

$$(9) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(10) \quad \nabla \times (\nabla f) = 0$$

$$(11) \quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

FUNDAMENTAL THEOREMS

$$\text{Gradient Theorem : } \int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

$$\text{Divergence Theorem : } \int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$$

$$\text{Curl Theorem : } \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$; $d\tau = dx dy dz$

$$\text{Gradient : } \quad \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence : } \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl : } \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian : } \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$; $d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient : } \quad \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{Divergence : } \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl : } \quad \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned}$$

$$\text{Laplacian : } \quad \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$; $d\tau = s ds d\phi dz$

$$\text{Gradient : } \quad \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence : } \quad \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl : } \quad \nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian : } \quad \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$