# PYL112: Minor II : Max marks 20: Time 1 hr.

Physics Department, I I T Delhi

21st March, 2015

All answers for a given question should preferably at one place. If mutiple answers are given for the same question then no credit will be given. All notations used for various operators etc. are standard and we omit ˆ sign.

1. An electron is moving in $x-y$ plane in presence of a magnetic field $B = B\hat{z}$. Consider the following form of the vector potential $A = \{0, Bx, 0\}$.
(a) Show that the energy eigenvalues of this quantum mechanical system is given by the Schrödinger equation of one dimensional simple harmonic oscillator along $x$-axis. Assuming a simple harmonic oscillator can be modelled as a mass-spring system write down the spring constant of this spring.
(b) Write down the general expression for energy eigenfunctions of the above system by comparing with the problem of one dimesional harmonic oscillator and plot the wavefunctions for the ground state and first excited states. Are these wavefunctions odd and even function of $x$.?
(c) Show that the above energy eigenvalues are degenerate and calculate this degeneracy. Explain the reason for these degeneracy.
$4 + 3 + 3$.

2. Consider the simultaneous eigenstates $|l, m\rangle$ of the $L^2$ and $L_z$ operators statisfying the eigenvalue equation

$$\begin{aligned} L^2|l,m\rangle &= l(l+1)\hbar^2|l,m\rangle \\ L_z|l,m\rangle &= m\hbar|l,m\rangle \end{aligned} \quad (1)$$

Consider a system with $l = 1$. Find the matrix representation of the operators $L_x$, $L_y$, $L_+$ and $L_-$ in the corresponding $|l, m\rangle$ basis. 6

3. Calculate the uncertainty product $\Delta x \Delta p_x$ for a one dimensional simple harmonic oscillators in a given energy eigenstate $|n\rangle$ using the properties of $a, a^\dagger$ operator. **Please note if you do this calculation using co-ordinate space wavefunction no marks will be given.** 4.

# PYL112: Major : Max marks 40: Time 2 hr.

Physics Department, I I T Delhi

30th April, 2015

All answers for a given question should preferably at one place. If mutiple answers are given for the same question then no credit will be given. All notations used for various operators etc. are standard.

1. (a) A hydrogen atom can be viewed as two point charge particles-a proton and an electron with Coulomb's interacting potential between them. Write the Schrödinger equation for such a system and separate it into two parts. One describing the motion of the center of the mass, and the other describing the relative motion of the proton and the electron. 5

   (b) In the class we have talked about hydrogen atom in three dimension ( real world). Now write down the Schrödinger equation for a two dimensional ( $x, y$ or $r, \phi$) hydrogen atom. Assume that the Coulomb interaction is of the form $-\frac{e^2}{r}$ where $r = \sqrt{x^2 + y^2}$. Now using separation of variables find the radial and the angular part equations. Solve the angular equations. Describe the quantum numbers that characterize the bound states and the degenracies of the system 5

2. (a) Consider two spin half particles ( example electron) whose spin operators are respectively given by $\boldsymbol{S}_1 = S_{1x}\hat{x} + S_{1y}\hat{y} + S_{1z}\hat{z}$ and $\boldsymbol{S}_2 = S_{2x}\hat{x} + S_{2y}\hat{y} + S_{2z}\hat{z}$. Ignore all other degrees of freedom except spin for these particles. Let the hamiltonian for this two-particle system is $H = -J\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$. Show that the eigenstates of this hamiltonian are simultaneous eigenstates of $\boldsymbol{S}_1^2, \boldsymbol{S}_2^2, \boldsymbol{S}^2, S_z$ operators where $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$ and $S_z$ is the $z$-component of $\boldsymbol{S}$. Find out these eigenstates using Clebsch-Gordon coefficients. 7.

   (b) Consider a wave fucntion for a hydrogen like atom:

$$\psi(r, \theta) = \frac{1}{81}\sqrt{\frac{2}{\pi}} Z^{\frac{3}{2}}(6 - Zr)Zr \exp(-\frac{Zr}{3}) \sin\theta \exp(i\phi)$$

   For such atom the nuclear charge is $Ze$. Here $r$ is in the unit of $a_0$, the Bohr radius. Find out the quantum number $n, l, m$ for this state with proper justification. 3.

3. (a) A full electron wave function is written as

$$\Psi = \Psi_{space} \otimes \Psi_{spin}$$

Electrons are fermions and hence the wavefunction is antisymmetric under two particle exchange. Followings are the candidate wavefunctions for two electrons. First part corresponds to the spatial degrees of freedom $(a, b)$and second part corresponds to the spin index. Identify which of these can be a two electron wavefunction with brief justification.

$$
\begin{aligned}
|\Psi^{\uparrow,\downarrow}_{a,b}\rangle &= (|a,b\rangle - |b,a\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|\Psi^{\uparrow,\downarrow}_{a,b}\rangle &= |(|a,b\rangle - |b,a\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\
|\Psi^{\uparrow,\downarrow}_{a,b}\rangle &= |(|a,b\rangle + |b,a\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|\Psi^{\uparrow,\downarrow}_{a,b}\rangle &= |(|a,b\rangle + |b,a\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\end{aligned}
$$

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(b) i. Why the spin-singlet state of two electrons is an entangled state? Consider a system made up of two spin $\frac{1}{2}$ particles. Observer $A$ specializes in measuring the spin components of one of the particles ($s_{1z}, s_{1x}$ and so on) while observer $B$ measures spin component of the other particle. Suppose the system is known to be in a spin singlet state with $S_{total} = 0$.
ii. What is the probability for observer $A$ to obtain $s_{1z} = \frac{\hbar}{2}$ when the observer $B$ makes no measuremnt ? Answer the same problem for $s_{1x}$.
iii. Observer $B$ determines the spin of the particle 2 to be $s_{2z} = \frac{\hbar}{2}$ with certainty. What can we then conclude about the observer $A$'s measuremt is (1) A measures $s_{1z}$ and (2) A measures $s_{1x}$?. Justify your answer. $2 + 2 + 2$.

4. (a) Spherical harmonics( $Y_{l,m}(\theta, \phi)$ or $|l, m\rangle$) are simultaneous eigenfunction of the operator $L^2$ and $L_z$ such that $L^2 Y_{l,m} = l(l+1)\hbar^2 Y_{l,m}$ and $L_z Y_{l,m} = m\hbar Y_{l,m}$ Now consider three such functions, namely, $Y_{1,1}, Y_{1,0}$ and $Y_{1,-1}$. Show that the simultaneous eigenfunctions of $L^2$ and $L_x$ are given by

$$
\begin{aligned}
\psi_1 &= \frac{1}{2}(Y_{1,1} + Y_{1,-1} + \sqrt{2} Y_{1,0}) \\
\psi_2 &= \frac{1}{\sqrt{2}}(Y_{1,1} - Y_{1,-1}) \\
\psi_3 &= \frac{1}{2}(Y_{1,1} + Y_{1,-1} - \sqrt{2} Y_{1,0})
\end{aligned}
$$

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(b) Consider a charged particle with charge $q$ is executing one dimensional simple harmonic motion along the $x$-axis. Apply an uniform electric field $\vec{E} = E\hat{x}$ on this particle. Write down the Hamiltonian and from there show that the particle will continue to execute one dimensional simple harmonic motion.

Now comparing with the problem of one dimensional simple harmonic oscillator discussed in the class find the energy eigenvalues and the ground state energy eigenfunction for the above problem. Please remember that you do not need to solve the Schrödinger equation for this. $2 + 3$.