# Department of Mathematics
# MTL 733 (Stochastic of Finance)
# Major Examination (I Semester 2017 - 2018)

Time allowed: 2 hours — Max. Marks: 40

Let $\{W(t), t \geq 0\}$ be a Wiener process and $\{N(t), t \geq 0\}$ be a Poisson process with parameter $\lambda$.

1. Define $X(t) = W(1-t) - W(1)$ for $t \in [0,1]$. Prove or disprove that $\{X(t), t \in [0,1]\}$ is a Brownian motion. (3 marks)

2. Let $X_1, X_2, \ldots$ be a i.i.d. random variables with $E(X_1) < \infty$. Define $S_n = X_1 + X_2 + \ldots + X_n$ and $\mathcal{F}_n = \sigma(S_n, S_{n+1}, \ldots)$. Prove or disprove that

$$E\left(\frac{1}{n} \mid \mathcal{F}_{n+1}\right) = \frac{1}{n} S_{n+1} \text{ almost surely}$$

(4 marks)

3. Evaluate $\int_0^t \left(\int_0^v dW(u)\right) dW(v)$, if it exists. (4 marks)

4. Consider two SDEs

$$dX(t) = a_1(X(t))dt + b_1(X(t))dW(t); \quad dY(t) = a_2(Y(t))dt + b_2(Y(t))dW(t)$$

Define $Z(t) = X(t)Y(t)$. Find $dZ(t)$. (4 marks)

5. Let $W^{(1)}(t)$ and $W^{(2)}(t)$ be two independent Brownian motions. Use Levy theorem to show that $M_t = \rho W^{(1)}(t) + \sqrt{1-\rho^2} W^{(2)}(t)$ is also a Brownian motion for a given constant $\rho$. (4 marks)

6. Let the interest rate $r$ and the volatility $\sigma > 0$ be constant. Let $S(t)$ be the stock price at any time $t$. Let

$$S(t) = S(0)e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

be a geometric Brownian motion with mean rate of return $r$, where the initial stock price $S(0)$ is positive and $\{W(t), t \geq 0\}$ is a Brownian motion. Let $K$ be the strike price at the maturity date $T$. Let $V(t)$ be portfolio at time $t$ and it satisfies

$$V(t) = a_t S(t) + b_t \beta(t), t \in [0, T].$$

Using change of measure, Show that, for $\theta = T - t$, the value of portfolio in the case of European put option is given by

$$V(t) = Ke^{-r\theta}\Phi(-z_2) - S(0)\Phi(-z_1)$$

where

$$z_1 = \frac{1}{\sigma\sqrt{\theta}}\left[\log\frac{S(0)}{K} + (r + 0.5\sigma^2)\,\theta\right], \quad z_2 = z_1 - \sigma\sqrt{\theta}$$

and $\Phi$ is the cumulative standard normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{1}{2}z^2}dz.$$

(4 marks)

$Y(t)$ ...
$+ X(t)\, \delta Y(t$

7. State and prove Feynman-Kac theorem. (2 + 3 marks)

8. Consider the jump diffusion model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + S(t^-)dJ(t)$$

where $J(t) = \sum_{j=1}^{N(t)} Y_j - \beta\lambda t$ is a compensated compound Poisson process, and $\beta = E(Y_i)$, where $Y_i$ are the jump sizes. Prove that,

$$S(t) = S(0)exp\left(\sigma W(t) + \left(\mu - \beta\lambda - \frac{\sigma^2}{2}\right)t\right)\prod_{j=1}^{N(t)}(Y_j + 1).$$

(4 marks)

9. Prove that $S(t) = e^{-\lambda\sigma t}(\sigma + 1)^{N(t)}$ is a martingale where $\sigma > -1$ is a constant. (4 marks)

10. Consider a Vasicek interest rate model

$$dr_t = \alpha(b - r_t)dt + \sigma dW_t,\ t \in [0, T],$$

where $\alpha, b$ and $\sigma$ are positive constants. Prove the algorithm for the exact simulation at time $0 = t_0 < t_1 < \ldots < t_n = T$ is given by

$$r_{t_{i+1}} = e^{-\alpha(t_{i+1}-t_i)}r_{t_i} + b\left(1 - e^{-\alpha(t_{i+1}-t_i)}\right) + \sigma\sqrt{\frac{1}{2\alpha}\left(1 - e^{-2\alpha(t_{i+1}-t_i)}\right)}\, Z_{i+1},\quad i = 0, 1, \ldots, n-1$$

where $Z_i$ is the standard normal sample. (4 marks)

$f_x = \sigma$ $S(t)$ D

$r_t = \sigma W(t) + k(t)$ $S(t)$ $(\mu - \beta\lambda)$ $= S(t) \cdot k'(t) + \frac{1}{2}\sigma^2 S$

$= \alpha(b - x_t)$ $\sigma$ $\partial x_t$ $= S_t$ $\mu - \beta\lambda -$

$k'(t)$ $= \alpha(b - \sigma W(t) - k(t))$ $\sigma W_t + k(t)$

$k(t)$ $k(t))\, k'(t) = S(t)\, (\mu - \beta\lambda)$